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On L_p-Convergence of Nonpolynomial Interpolation

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1. INTRODUCTION

During the last fifty years considerable literature has grown around the problem of weighted mean convergence of Lagrange interpolation polynomials on the zeros of certain orthogonal polynomials with a non-negative weight function $w(x)$. This topic was first initiated by Erdös and Turán in a classic paper [3]. Denoting the Lagrange interpolant of degree n to f by $L_n^f(x)$, Erdös and Turán proved that if $w(x)>0$ on (a, b) and $w \in L_1[a, b]$, then

$$\lim_{n \to \infty} \int_a^b |L_n^f(x) - f(x)|^2 \, w(x) \, dx = 0. \tag{1}$$

Later Erdös and Feldheim [2] showed that, for $L_n^f(x)$ based on the zeros of Tchebycheff polynomials of the first kind, and with $w(x) = (1-x^2)^{-1/2}$, a stronger result holds, viz.,

$$\lim_{n \to \infty} \|L_n^f - f\|_p^p := \lim_{n \to \infty} \int_a^b |L_n^f(x) - f(x)|^p \, w(x) \, dx = 0,$$
$$p = 1, 2, ... \tag{2}$$

On the other hand, if $L_n^f(x)$ is based on the zeros of Tchebycheff polynomials of the second kind and if $w(x)=1$, there exists a continuous function f for which $\lim_{n \to \infty} \int_a^b |L_n^f(x) - f(x)|^2 \, dx = \infty$.

R. Askey [1] has examined the problem in depth and has considered the case where the nodes are the zeros of Jacobi polynomials, $w(x) = (1-x)^\alpha (1+x)^\beta$, $\alpha, \beta \geqslant -\frac{1}{2}$. More recently P. Nevai [9] has gone further and has obtained necessary and sufficient conditions for mean convergence of Lagrange and quasi-Lagrange interpolation based on the

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zeros of generalized Jaobi polynomials. For other similar results we refer to Feldheim [4, 5], Nevai [8], and Varma and Vertesi [12].

Here we shall be interested in this problem from a new angle recently introduced by I. H. Sloan [10]. He considered the eigenvalue problem

$$p(x)\,u''(x)+q(x)\,u'(x)+(\lambda+r(x))\,u(x)=0 \tag{3}$$

with the boundary conditions

$$\cos\alpha u(a)+\sin\alpha u'(a)=0, \qquad \cos\beta u(b)+\sin\beta u'(b)=0, \tag{4}$$

where $p\in C^2[a,b]$, $q\in C^1[a,b]$, $r\in C[a,b]$, $p(x)>0$, and $q(x)$, $r(x)$, α, β are all real. Let $\{u_j\}_0^\infty$ be the eigenfunctions arranged according to increasing eigenvalues λ_j and let $L_n^f(x)$ be the unique linear combination of $u_0, ..., u_n$ that coincides with f at the $(n+1)$ zeros of $u_{n+1}(x)$ that lie in the open intervals (a,b). Further let

$$w(x)=\frac{1}{p(x)}\exp\left(\int_a^x \frac{q(t)}{p(t)}\,dt\right). \tag{5}$$

He has shown that the limit (1) holds for all $f\in C[a,b]$, provided f satisfies $f(a)=0$ if $\sin\alpha=0$ and $f(b)=0$ if $\sin\beta=0$. Moreover, $\|L_n^f-f\|_p\leqslant CE_n(f)$ where $E_n(f)$ is the error of best uniform approximation to f by a linear combination of $u_0, ..., u_n$ and C is a constant.

In this paper we adapt the method of Sloan to show that under the hypotheses (3), (4), and (5), the mean convergence holds even in the L_p-norm, $p\geqslant 1$. Moreover,

$$\|L_n^f-f\|_p\leqslant C_nE_n(f), \qquad p\geqslant 1, \tag{6}$$

where $C_n=O(n^{1-1/2p})$ and $f\in \mathrm{Lip}[a,b]$.

2. Preliminaries and Main Result

Let $\{u_i\}_{i=0}^\infty$ be the eigenfunctions of (3) and (4) arranged according to increasing eigenvalues $\{\lambda_j\}_{j=0}^\infty$.

As it stands, the boundary-value problem is not in self-adjoint form, but it becomes so on multiplying (3) by the integrating factor $w(x)$ defined by (5). It then follows from the classical theory that the eigenvalues are real and have their only accumulation point at $+\infty$, and that the eigenfunctions $u_0, u_1, ...$ are uniquely determined apart from a multiplicative factor, and are orthogonal with respect to the weight function $w(x)$.

Moreover, it follows from the work of Gantmacher and Krein [7, pp. 33–36] that $u_{n+1}(x)$ has exactly $n+1$ zeros $x_0, x_1, ..., x_n$ in the open interval (a, b), and also that the matrix $\{u_i(t_j)\}_{i,j=0}^n$ has nonzero determinant if $t_0, ..., t_n$ are any distinct points in (a, b). From these two properties it follows that the interpolant $L_n^f = \sum_0^n a_j u_j$, which interpolates f at the zeros of $u_{n+1}(x)$, exists and is unique for every value of n.

Let U_n denote the finite dimensional subspace spanned by $\{v_j\}_{j=0}^n$, where

$$v_j = \frac{1}{\|u_j\|_2} u_j,$$

$$b$$

then $(v_i, v_j) = \delta_{ij}$, $0 \leqslant i, j \leqslant n$. Setting $K_n(x, y) := \sum_{i=0}^n v_i(x)\, v_i y)$ and putting

$$\begin{aligned} \lambda_i(x) &= \frac{K_n(x_i, x)}{K_n(x_i, x_i)}, \qquad i = 0, ..., n, \\ \Lambda_{ij} = \lambda_i(x_j) &= \frac{K_n(x_i, x_j)}{K_n(x_i, x_i)}, \qquad i, j = 0, 1, ..., n, \end{aligned} \tag{7}$$

we state our main results.

THEOREM 1. *Suppose that for n sufficiently large we have*

$$\sum_{\substack{i=0 \\ i \neq j}}^{n} \frac{|K_n(x_i, x_j)|}{K_n(x_i, x_i)} \leqslant \rho < 1, \qquad j = 0, ..., n, \tag{8}$$

and

$$\sum_{j=0}^{n} \|\lambda_j(\cdot)\|_p = c_n^*, \qquad p \geqslant 2, \tag{9}$$

where ρ *is a constant and* $c_n^* = O(n^{1-1/p})$. *Then* L_n^f *exists and is unique for n sufficiently large and the estimate* (6) *holds.*

THEOREM 2. *Under the hypotheses related to the boundary-value problem* (3), (4), *and* (5), *let* $L_n^f = \sum a_i u_i$ *be the interpolant to* f *on zeros of* $u_{n+1}(x)$. *Then* (2) *holds for all* $f \in \mathrm{Lip}[a, b]$, *provided* $f(a) = 0$ *if* $\sin \alpha = 0$ *and* $f(b) = 0$ *if* $\sin \beta = 0$. *Moreover,* (6) *holds.*

The proofs of Theorem 1 and 2 are given in Sections 3 and 4. In order to prove Theorem 1 we need the following Lemma.

LEMMA 1. *Under the conditions of Theorem* 1, *we have*

$$\|L_n^f - f\|_p \leqslant (\|L_n\|_p + M)\, E_n(f),$$

for all $f \in C[a,b]$, *where* $\|L_n\|_p = \sup_{f\in C[a,b]} \|L_n^f\|_p/\|f\|_\infty$, $M = (\int_a^b w(x)\,dx)^{1/p}$ *and* $0<p<\infty$.

Proof. For $u \in U_n$, we have

$$\|L_n^f - f\|_p = \|L_n^{f-u} - (f-u)\|_p \leqslant \|L_n^{f-u}\|_p + \|f-u\|_p$$
$$\leqslant \|L_n\|_p \, \|f-u\|_\infty + M\|f-u\|_\infty.$$

Since u is an arbitrary element of U_n,

$$\|L_n^f - f\|_p \leqslant (\|L_n\|_p + M)\, E_n(f). \quad \blacksquare$$

To prove Theorem 2, we first prove it for the simple differential equation when $p(x)=1$ and $q(x)=0$, i.e.,

$$u''(x) + [r(x)+\lambda]\,u(x) = 0 \tag{10}$$

and the boundary conditions (4). Then we extend the result to the more general situation. To prove Theorem 2 for this special case (10) we need the following Theorems A and B of Jackson [6, pp. 449, 453].

If f has the series expansion

$$f(x) = \sum_{i=0}^{\infty} \alpha_i u_i(x), \qquad \alpha_i = \int_0^\pi f(u_i(x)\,dx \bigg/ \int_0^\pi u_i^2(x)\,dx,$$

and if $\sigma_n(x) := \sum_{i=0}^n \alpha_i u_i(x)$, then we have the following Theorems of Jackson.

THEOREM A. *If* $f(x)$ *has a continuous* kth *derivative of bounded variation in the interval* $0 \leqslant x \leqslant \pi$, *while* f *itself and its* $(k-1)$ *derivatives vanish for* $x=0$ *and for* $x=\pi$, *and if, furthermore,* $r(x)$ *has a continuous* $(k-2)$*th derivative of bounded variation in* $0\leqslant x\leqslant \pi$, *then*

$$f(x) = \sigma_n(x) + O\left(\frac{1}{n^k}\right)$$

uniformly throughout the interval.

For the special case $k=1$, the theorem is true without the restriction $f(0)=f(\pi)=0$.

THEOREM B. *If* $f(x)$ *satisfies the Lipschitz condition*

$$|f(x_1) - f(x_2)| \leqslant \mu\,|x_1 - x_2|$$

throughout the interval $[0, \pi]$, *and if* $f(0)=0$, *then in the whole interval*

$$|f(x)-\sigma_n(x)| \leqslant \frac{c\mu \log n}{n}, \qquad n \geqslant 2,$$

where c is independent of x, n, and f. The restriction on $r(x)$ *in* (10) *is merely that of continuity.*

3. Proof of Theorem 1

We shall show that the nth order matrix $\Lambda=(\Lambda_{ij})$ is nonsingular if n is sufficiently large, from which it will follow that L_n^f exists and is unique for large n.

If $|||\cdot|||$ denotes the matrix norm $|||\Lambda||| = \max_{0 \leqslant j \leqslant n} \sum_{i=0}^{n} |\Lambda_{ij}|$, then Condition (8) is equivalent to $|||\Lambda - I||| \leqslant \rho < 1$, where I is the unit matrix of order $n+1$. From this it follows that Λ is nonsingular, and in fact $|||\Lambda^{-1}||| \leqslant (1-\rho)^{-1}$, so that the inverses are bounded independently of n.

Now set

$$l_i(x) := \sum_{j=0}^{n} (\Lambda^{-1})_{ij}\, \lambda_j(x), \qquad i=0, ..., n. \tag{11}$$

Then it is easily verified that $l_i(x_j)=\delta_{ij}$, and from this it follows that the interpolating approximation L_n^f is given by

$$L_n^f(x) = \sum_{i=0}^{n} l_i(x)\, f(x_i). \tag{12}$$

Then on using an argument analogous to that of Sloan, (11) and (12) yield

$$\begin{aligned}
\|L_n^f\|_p^p &= \int_a^b |L_n^f(x)|^p\,dx = \int_a^b \left|\sum_{i=0}^{n} f(x_i)\, l_i(x)\right|^p dx \\
&= \int_a^b \left|\sum_{i=0}^{n} f(x_i) \sum_{j=0}^{n} (\Lambda^{-1})_{ij}\, \lambda_j(x)\right|^p dx \\
&= \int_a^b \left|\sum_{j=0}^{n} \left(\sum_{i=0}^{n} f(x_i)(\Lambda^{-1})_{ij}\right) \lambda_j(x)\right|^p dx \\
&\leqslant (\|f\|_\infty\, |||\Lambda^{-1}|||)^p \int_a^b \left(\sum_{j=0}^{n} |\lambda_j(x)|\right)^p dx.
\end{aligned}$$

By the Minkowski inequality and (9), we have

$$\|L_n\|_p \leqslant (1-\rho)^{-1} c_n^* = O(n^{1-1/p}),$$

and the remainder of the theorem now follows from Lemma 1.

Remark. For $p=2$ Sloan has the advantage that he can exploit the orthogonal property $(v_i, v_j)=\delta_{ij}$ and can thereby find a stronger bound. For $p \neq 2$ this is not applicable any more and so the result of Sloan cannot be recovered by setting $p=2$.

4. Proof of Theorem 2.

As mentioned earlier, we first prove the result for the simple differential equation (10) and the boundary condition (4). For this case, the weight function defined by (5) reduces to $w(x) \equiv 1$, so that the inner product becomes $(u, v) = \int_a^b u(x)\, v(x)\, dx$.

If we assume for convenience that the eigenfunctions are normalized by $(u_i, u_i) \equiv \|u_i\|_2^2 = 1$, then the orthogonality relation for the eigenfunctions becomes $(u_i, u_j) = \delta_{ij}$, $i, j = 0, 1, \ldots$. Consequently, the kernel $K_n(x, y)$ can be written as

$$K_n(x, y) = \sum_{i=0}^{n} u_i(x)\, u_i(y),$$

which is a symmetric function of x and y.

For Theorem 1 to be applicable we must show that $K_n(x, y)$ and $\lambda_j(x)$, $j = 0, 1, \ldots, n$, satisfy Conditions (8) and (9), where $x_0, \ldots, x_n$ are the zeros of $u_{n+1}(x)$. Sloan has obtained asymptotic expressions for $K_n(x, y)$ and $K_n(x, x)$ by means of contour integral methods employed by Titchmarsh to study Sturm–Liouville series. He has also found asymptotic estimates for the zeros of $u_{n+1}(x)$. He has shown that for the interpolation points, we have

$$K_n(x_i, x_i) = \frac{n}{b-a} + O(1)$$

and

$$\max_{0 \leqslant j \leqslant n} \sum_{\substack{i=0 \\ i \neq j}}^{n} \frac{|K_n(x_i, x_j)|}{K_n(x_i, x_i)} = O\left(\frac{\log n}{n}\right).$$

Therefore Condition (8) of Theorem 1 is satisfied if n is large enough. We now aim to show that Condition (9) is also satisfied.

LEMMA 2. *The following estimate holds for $p \geqslant 2$:*

$$\sum_{j=0}^{n} \|\lambda_j(\cdot)\|_p = c_n^* = O(n^{1-1/p}).$$

Proof. We shall work out in detail the case $\sin\alpha = \sin\beta = 0$. A similar argument yields the result for the cases $\sin\alpha \neq 0$, $\sin\beta = 0$ and $\sin\alpha \neq 0$, $\sin\beta \neq 0$. If we set

$$D_n(x) = \frac{\sin(n+\frac{1}{2})\,x}{\sin(x/2)},$$

then for $x \neq x_j$, we have

$$K_n(x, x_j) = K_n(x_j, x) = I_{1,j} + I_{2,j},$$

where

$$I_{1,j} = \frac{1}{b-a}\left[D_{n1}\left(\pi\frac{x_j - x}{b-a}\right) - D_{n+1}\left(\pi\frac{x + x_j - 2a}{b-a}\right)\right]$$

and

$$I_{2,j} = \frac{1}{2\pi i}\int_C O\left(\frac{1}{|s|^2}e^{-|t|\,|x_j - x|}\right)d\lambda,$$

with $\lambda = s^2$, $s = \sigma + it$. Here the contour C is the boundary of the rectangle $0 \leqslant \sigma \leqslant (n+\frac{3}{2})\,\pi/(b-a)$, $|t| \leqslant (n+\frac{3}{2})\,\pi/(b-a)$.

Let $\alpha_j = \pi((x - x_j/(b-a))$ and $\beta_j = \pi((x + x_j - 2a)/(b-a))$. Since

$$I_{1,j} = \frac{1}{(b-a)}\left[\frac{\sin(n+2)\,\alpha_j}{2\sin(\alpha_j/2)}\cos\frac{\alpha_j}{2} - \frac{\sin(n+2)\,\beta_j}{2\sin(\beta_j/2)}\cos\frac{\beta_j}{2} + \frac{1}{2}(\cos(n+2)\,\beta_j - \cos(n+2)\,\alpha_j)\right],$$

we have

$$|I_{1,j}| \leqslant \frac{1}{b-a}\left(\left|\frac{\sin(n+2)\,\alpha_j}{2\sin\alpha_{j/2}}\right| + \left|\frac{\sin(n+2)\,\beta_j}{2\sin(\beta_j/2)}\right| + 1\right), \quad j = 0, \ldots, n. \tag{13}$$

Now we show $I_{2,j} = O(1)$ uniformly for $a \leqslant x \leqslant b$. Thus we have

$$|O(|s|^{-2}e^{|t|(x-a)})| \leqslant c_1\,|s|^{-2}\,e^{|t|(x-a)}, \qquad a \leqslant x \leqslant b,$$

where c_1 is independent of x and s. Therefore we have

$$|O(|s|^{-2} e^{-|t||x_j-x|})| \leqslant |O(|s^{-2}|)| \leqslant c_1 |s|^{-2}.$$

Now $I_{2j} := \sum_{K=1}^{6} I_K$ where

$$I_K := \int_{c_K} O\left(\frac{1}{|s|^2} e^{-|t||x_j-x|}\right) d(s^2),$$

where c_K's form the boundary of the rectangle in Fig. 1. From the above estimates, it is easy to see that $I_3 + I_5 = 0$, $|I_2 + I_6| \leqslant c_1/2$, $|I_1| \leqslant 2c_1(\ln(1+\sqrt{2})/\pi)$, and $|I_4| \leqslant c_1$. Combining these estimates, we see that $I_{2,j} = O(1)$.

Combining this and (13), we then have

$$|K_n(x_j, x)| \leqslant \frac{1}{(b-a)} \left(\left| \frac{\sin(n+2)\,\alpha_j}{2 \sin \alpha_j/2} \right| + \left| \frac{\sin(n+2)\,\beta_j}{2 \sin \beta_j/2} \right| \right) + O(1)$$

and hence

$$\|K_n(x_j, \cdot)\|_p \leqslant \frac{1}{(b-a)} \left[\left(\int_a^b \left| \frac{\sin(n+2)\,\alpha_j}{2 \sin \alpha_j/2} \right|^p dx \right)^{1/p} + \left(\int_a^b \left| \frac{\sin(n+2)\,\alpha_j}{2 \sin \beta_j/2} \right| dx \right)^{1/p} + O(1) \right].$$

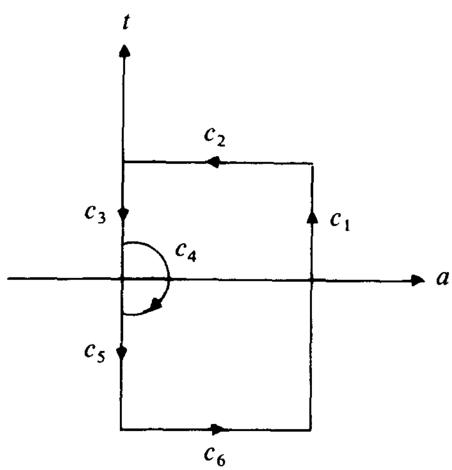

FIGURE 1

It is easy to see that

$$\int_a^b \left|\frac{\sin(n+2)\,\alpha_j}{2\sin\alpha_j/2}\right|^p dx = \frac{(b-a)}{\pi}\int_{\pi(a-x_j)/(b-a)}^{\pi(b-x_j)/(b-a)} \left|\frac{\sin(n+2)\,u}{2\sin(u/2)}\right|^p du$$

$$\leqslant \frac{(b-a)}{\pi}\int_{\pi(a-x_j)/(b-a)}^{\pi(1+(b-x_j)/(b-a))} \left|\frac{\sin(n+2)\,u}{2\sin u/2}\right|^p du$$

$$= \frac{(b-a)}{\pi}\int_{-\pi}^{\pi} \left|\frac{\sin(n+2)\,u}{2\sin u/2}\right|^p du$$

but since $|\sin(n+2)\,u| \leqslant 2(n+2)\,|\sin(u/2)|$, it follows that

$$\int_{-\pi}^{\pi} \left|\frac{\sin(n+2)\,u}{2\sin u/2}\right|^p du \leqslant (n+2)^{p-2}\int_{-\pi}^{\pi} \left|\frac{\sin(n+2)\,u}{2\sin u/2}\right|^2 du$$

$$= \pi(n+2)^{p-1}, \qquad p \geqslant 2.$$

Similarly, we have

$$\int_a^b \left|\frac{\sin(n+2)\,\beta_j}{2\sin\beta_j/2}\right|^p dx \leqslant \pi(n+2)^{p-1}, \qquad p \geqslant 2.$$

Therefore,

$$\|K_n(x_j, \cdot)\|_p \leqslant 2(b-a)^{1/p}(n+2)^{1-1/P} + O(1)$$

and hence

$$\sum_{j=0}^{n} \|\lambda_j(\cdot)\|_p = \sum_{j=0}^{n} \left\|\frac{K_n(x_j, \cdot)}{K_n(x_j, x_j)}\right\|_p = O(n^{1-1/p}), \qquad p \geqslant 2. \quad \blacksquare$$

Now the proof of Theorem 2 in the special case (10) follows from the fact

$$E_n(f) \leqslant |f(x) - \sigma_{n,1}(x) - \sigma_{n,2}(x)|$$

$$\leqslant |f(a) - \sigma_{n,1}(x)| + |f(x) - f(a) - \sigma_{n,2}(x)|,$$

where $\sigma_{n,1}(x)$ and $\sigma_{n,2}(x)$ are the corresponding sums formed for the constant function $f(a)$ and $f(x) - f(a)$, respectively, and hence

$$E_n(f) \leqslant O\left(\frac{1}{n}\right) + c\,\frac{\log n}{n}.$$

Therefore,

$$\left[\int_a^b |L_n^f(x)-f(x)|^p\,dx\right]^{1/p} \leqslant c_n E_n(f) = O(n^{1-1/p})\left(O\left(\frac{1}{n}\right)+c\,\frac{\log n}{n}\right)$$

which converges to 0 as $n \to \infty$. We then have (2) and (6).

We now extend the result to the more general boundary value problem defined by (3) and (4). The proof is similar to the proof given by Sloan for the case $p=2$. With a similar argument we obtain, for $p \geqslant 2$,

$$\lim_{n\to\infty} \int_a^b |L_n^f(x)-f(x)|^p\,w_p(x)\,dx = 0, \tag{14}$$

where $w_p(x)=p(x)^{(p-2)/4}\,w(x)^{p/2}$ and also $[\int_a^b |L_n^f(x)-f(x)|^p\,w_p(x)\,dx]^{1/p} \leqslant c_n E_n(f)$ where $c_n=O(n^{1-1/p})$. The proof of Theorem 2 stated in Section 1 can be immediately obtained from (14), viz.

$$\int_a^b |L_n^f(x)-f(x)|^{p/2}\,w(x)\,dx$$
$$\leqslant \left(\int_a^b |L_n^f(x)-f(x)|^p\,w_p(x)\,dx\right)^{1/2}\left(\int_a^b \frac{w^2(x)}{w_p(x)}\,dx\right)^{1/2}.$$

The result also holds if $w(x)\equiv 1$. However, the weight function given by (5) is in a sense the natural weight function for this problem.

As an application of this proposition consider the following eigenvalue problem, based on the Bessel equation

$$u''(x)+\frac{1}{x}\,u'(x)+\left(\lambda-\frac{v^2}{x^2}\right)u(x)=0, \qquad 0<a<x=b,$$

with boundary conditions $u(a)=u(b)=0$. The eigenvalues $\lambda_n=s_n^2$ are determined by $J_v(s_nb)\,Y_v(s_na)-Y_v(s_nb)\,J_v(s_na)=0$ and an eigenfuncton u_n corresponding to λ_n is $u_n(x)=J_v(s_nx)\,Y_v(s_na)-Y_v(s_nx)\,J_v(s_na)$. Then we have (2) and (6) for all $f\in \mathrm{Lip}[a,b]$.

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